

Two Party Non-Local Games

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Abstract In this work we have introduced two party games with respective winning conditions. One cannot win these games deterministically in the classical world if they are not allowed to communicate at any stage of the game. Interestingly we find out that in quantum world, these winning conditions can be achieved if the players share an entangled state. We also introduced a game which is impossible to win if the players are not allowed to communicate in classical world (both probabilistically and deterministically), yet there exists a perfect quantum strategy by following which, one can attain the winning condition of the game.

Keywords Non-local games · Quantum strategy · Classical strategy

1 Introduction

It is well known that in quantum information theory the amount of communication required to perform various distributed computational tasks is much less than its classical protocol [1, 2]. Given a prior share of entanglement between two spatially separated parties, the amount of classical communication required for distributed computational tasks can be reduced to a large extent. Inspired by the nonlocal properties of the entangled states various pseudo telepathy games were introduced. The study of pseudo telepathy games was first introduced in [2], although the term pseudo-telepathy was first introduced in [6]. Later a multi party pseudo-telepathy game was introduced [4]. The first explicit quantum pseudo telepathy game was proposed by Vaidman [3] as a minor variation of [5]. Indeed, Vaidman was also the first to suggest [11] the first two-player quantum pseudo telepathy game as a variation of Cabello's proof [12, 13]. The well known Pseudo-telepathy games are 'Impossible Colouring Game' [5, 6], 'Parity Game' [4, 6], 'Deutsch-Jozsa Game' [3, 6], 'Matching

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Game' [7], 'Magic Square game' [8, 9] and so many. A different two-player quantum pseudo telepathy game has been recently introduced in [14].

In order to formulate a multi party pseudo telepathy game let us consider n parties $X_1, X_2, X_3, \dots, X_n$ and two n -ary functions f and g . Initially they are allowed to discuss their strategy and game plan (in classical situation) and also entanglement for quantum case. Once they are separated, they are strictly prohibited from any further communication. After that they are provided with inputs a_i and asked to produce the output b_i . The players are said to be winners if they satisfy the winning condition $f(a_1, a_2, \dots, a_n) = g(b_1, b_2, \dots, b_n)$. For a particular game we define a n ary predicate P known as the promise. We say that the protocol is perfect if the promise is satisfied. There are several reasons for which pseudo-telepathy games are appealing to us.

- (1) Conceptually these games are very simple and comprehensible.
- (2) If these games are carried out successfully, these will provide strong evidence of the fact that quantum physics can be harnessed to reduce the amount of communication required to perform various distributive works.

However there are certain features for which the experimental demonstration of pseudo telepathy games are not appealing to the physicists.

- (1) A small margin of error in the implementation of the game would result in very high error probability than the corresponding classical protocol.
- (2) The evidence of convincing quantum behavior would require a large number consecutive successful runs.

In this work we introduced few two party non local games. These games are different from pseudo-telepathy games in the sense that, these games unlike pseudo-telepathy games do not allow any kind of communication between the players before the onset of the games. It is a well known fact that pseudo telepathy games do not exist for bi-partite entangled states [10]. Classical players, without communicating with one another, cannot achieve the winning condition of these games deterministically, where as the quantum players with prior share of the entanglement can exactly do so. In this work we claim to develop a two party game which is impossible to win in the classical world (both deterministically and probabilistically), and interestingly we find a quantum strategy by which this game can be won with a prior share of entanglement between these two players.

2 Two Party Non-Local Games: Perfect Quantum Strategies

Game 1 Let us consider two players Alice and Bob who are separated from each other in such a way so that they cannot communicate with each other. Before these two players are separated they are not allowed to communicate between each other so that they can share random variables and also not allowed to discuss among themselves to determine a particular strategy to win the game. Once they are separated far apart they are devoid of any kind of communication between each other. At a certain instant the two spatially separated parties are asked a question \mathbf{X} , which has two possible answers $\{+1, -1\}$. Interestingly the same question will be asked to both players for n times. Not only that each of the players are deprived of the memory of the answers given by them in previous trials. This game has no such promise which has to be fulfilled by the inputs or in other words no such restrictions are given on the legitimacy of the question asked to both the players. The players are said to

win the game if the sum of all the outputs given by both the players at the end of all the n possible trials adds to zero. Mathematically, the winning condition is given by,

$$\sum_i^n (X_i^A + X_i^B) = 0 \tag{1}$$

where X_i^A and X_i^B are the answers provided by Alice and Bob at the i th trial respectively. In classical setting the game cannot be won deterministically. However the winning condition can be achieved by the classical players at least probabilistically.

If $n = 1$, the total number of outcomes are 4 and the probability of obtaining the sum 0 is $\frac{1}{2}$. All possible outcomes are given by,

$$\begin{aligned} +1 - 1 &= 0, \\ +1 + 1 &= 2, \\ -1 - 1 &= 2, \\ -1 + 1 &= 0. \end{aligned} \tag{2}$$

If $n = 2$ all possible outcomes of the game are 16. The number of cases favorable to the event of obtaining the sum of the answers to be 0 are,

$$\begin{aligned} +1 + 1 - 1 - 1 &= 0, \\ +1 - 1 + 1 - 1 &= 0, \\ -1 + 1 + 1 - 1 &= 0, \\ +1 - 1 - 1 + 1 &= 0, \\ -1 + 1 - 1 + 1 &= 0, \\ -1 - 1 + 1 + 1 &= 0. \end{aligned} \tag{3}$$

It is clearly evident from (3) the probability for winning the game classically is $\frac{6}{16}$. A little mathematical calculation will show that for n trials, the total number of outcomes are 2^{2n} . The number of cases which are favorable to the event of winning the game are $2^n C_n$. Hence the probability of obtaining the sum answers given by two parties after n trials as 0 is $\frac{2^n C_n}{2^{2n}}$. As n tends to a large number the probability of the event tends to 0, i.e $\lim_{n \rightarrow \infty} \frac{2^n C_n}{2^{2n}} = 0$. So it is clearly evident that the game described above cannot be done in the classical world deterministically.

However there exists a perfect quantum solution to the game. Initially Alice and Bob are allowed to share an entangled state and decide the quantum strategy so that their outputs can attain the winning condition of the game. Let us assume that Alice and Bob share an entangled state,

$$|\psi\rangle = \frac{1}{\sqrt{2}}[|01\rangle - |10\rangle] \tag{4}$$

(where the first qubit belongs to Alice whereas the second qubit belongs to Bob). The strategy to win the game is like this: If a member is asked the **X** question, he/she will measure the spin of his/her qubit along x direction, i.e he/she will measure σ_x . Measurement results

(+1 and −1 eigenvalues) will be the answers. Now it is clearly evident that the state $|\psi\rangle$ has an eigen value 0 for the observable $\sum_{i=1}^n (\sigma_{x,i}^A + \sigma_{x,i}^B)$ (where i denotes the number of trials),

$$\sum_{i=1}^n (\sigma_{x,i}^A + \sigma_{x,i}^B)|\psi\rangle = 0|\psi\rangle. \tag{5}$$

Hence we can conclude that they can win the game in every possible cases. Thus we see that for above described game it is impossible for the classical players to win the game deterministically, whereas there exists a quantum strategy for quantum players to achieve the winning condition.

Game 2 In this section we will consider another non-local game, impossible to win deterministically in the classical world, but still admits of a quantum solution. Let us consider two friends Alice and Bob who are not initially allowed to interact so that they can share whatever information they like to. Even when they are separated, they are not allowed to communicate any kind of information between them. Once again each of them are asked a question \mathbf{X} for n number of times. The solutions to the question are $\{+1, -1\}$. Each of the player’s answers in any particular trial doesn’t depend on what they have answered in previous trials. The game has been so designed that the players are declared as the winners only when the product of the outputs of the two players at the end of n trials is equal to 1. Mathematically, the winning condition is given by,

$$\prod_{i=1}^n X_i^A X_i^B = 1 \tag{6}$$

where X_i^A and X_i^B are the outcomes of Alice and Bob at the i th trial respectively. Classically the distant players do not have a deterministic solution to the above game. But still one can win the game with certain probabilities of success. A straight forward observation will reveal that the probability obtaining the product of the outcomes by two players at the end of n trials to be 1 is $\frac{1}{2}$. Whatever be the number of trials, the probability will not decrease like the previous game and will remain as a constant quantity $\frac{1}{2}$.

However there exists a quantum strategy under which the winning condition can be achieved by the players deterministically if they share an entangled state initially. The quantum players are not allowed to communicate with each other at any stage of the game. Let us consider two parties Alice and Bob share an entangled state of the form

$$|\psi'\rangle = \frac{1}{\sqrt{2}}[|00\rangle + |11\rangle]. \tag{7}$$

If once again one can think of that the asking the question \mathbf{X} to one of the players is equivalent of measuring the spin of the qubit along x direction (i.e he/she will measure σ_x), then the answers to the question are the measurement results +1 and −1. If $\prod_{i=1}^n \sigma_{x,i}^A \sigma_{x,i}^B$ is the operator to be applied to the state $|\psi'\rangle$, we obtain,

$$\prod_{i=1}^n \sigma_{x,i}^A \otimes \sigma_{x,i}^B |\psi'\rangle = |\psi'\rangle \tag{8}$$

(where i denotes the number of trials). This clearly indicates that the operator $\prod_{i=1}^n \sigma_{x,i}^A \sigma_{x,i}^B$ on its action on the state $|\psi'\rangle$ gives the state back with eigen value 1, satisfying the winning condition in all possible cases.

Game 3 This is the most important of all the two party non local games discussed so far. The speciality of the game lies in its impossibility in the classical settings not only deterministically but even probabilistically, but still we can find a quantum strategy by which the game can be won. Quite alike to the previous cases here we have also considered two players Alice and Bob who are not allowed to communicate at any stage of the game. The promise of the game is that if one of the players are asked to answer the question \mathbf{X} , the other will be asked the question $\bar{\mathbf{X}}$ which is just the complementary question to \mathbf{X} . The solution to both the questions \mathbf{X} and $\bar{\mathbf{X}}$ is taken from the set $G = \{1, -1\}$. The elements 1 and -1 are additive inverse of each other in the group $(\mathbf{R}, +)$, where $+$ is the normal arithmetical addition. Without any loss of generality let us assume that Alice was asked to answer the question \mathbf{X} and Bob was asked to answer the question $\bar{\mathbf{X}}$. Now Alice can choose the answer from any two elements of the set G (as both of them are feasible answers to the question \mathbf{X}). Similarly Bob can also choose the answer of his question $\bar{\mathbf{X}}$ from the same set G . The players are said to be winners of the game if the product of their answers is equal to 1, provided that answer given by one player is the additive inverse of the answer given by another player. Mathematically, the winning condition can be written as,

$$X^A \bar{X}^B = 1 \quad (9)$$

where the output \bar{X}^B of Bob is the additive inverse of the output X^A given by A. The game is interesting in the sense that the winning condition is set up in such a way that it is impossible to achieve in the classical world, even if they are allowed to communicate. This is because of the fact this is an impossible event, the situation will never arise when the product of two additive inverses will be equal to 1.

However quantum mechanically, with a prior share of quantum entanglement this can be easily achieved. Let us consider two players Alice and Bob are sharing an entangled state of the form,

$$|\psi'\rangle = \frac{1}{\sqrt{2}}[|00\rangle + |11\rangle]. \quad (10)$$

Now if we consider the asking the question \mathbf{X} to the quantum players is equivalent of saying that they are measuring the spin of their qubit along x direction, i.e he/she will measure σ_x , then in quantum mechanical context the complementary question $\bar{\mathbf{X}}$ will be the measuring same σ_x . This is because of the fact $\sigma_x^2 = I$ (where I is the identity operator). Measurement result ($+1$ and -1 eigen values) will be the answer. It is clearly evident already from the previous section that

$$\sigma_x^A \otimes \sigma_x^B |\psi'\rangle = |\psi'\rangle. \quad (11)$$

It is evident from the above equation the winning condition is always achieved, even when the eigen values of the respective operators are additive inverse of each other. Thus we see that this particular game attains the winning condition quantum mechanically, even when it is impossible to do so in the classical setting.

3 Conclusion

In summary we can say that here in this work we proposed three two party non local games, which doesn't admit of a winning solution in the classical world at least deterministically.

Interestingly players can win this game if they share a prior entangled state in between them. These games are different from the existing pseudo-telepathy games in the sense that these games do not involve any kind of projective measurements in the computational basis. Not only that for these games we have used entangled states of dimension 2×2 . However for pseudo telepathy games this is not possible [10]. Also for these non local games we not allowing classical players to communicate at any stage of the game. However in pseudo-telepathy games classical players are allowed to communicate before the onset of the game. In the first two games one can win the game not deterministically, but with a certain probability of success. Interestingly this probability of success is small even when the input is very small. However the third game is different from the first two in the sense that there doesn't exist any classical setting by which one can win the game, but with a prior share of entanglement, the quantum players will be successful in winning the game in all possible situations. One can also implement the concept of these non local games introduced in this work in multi party settings.

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